# XIV. MICROWAVE THEORY

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## A. ANALYSIS OF BILATERAL, TWO TERMINAL-PAIR NETWORKS IN THE THREE-DIMENSIONAL HYPERBOLIC SPACE

In the Quarterly Progress Report of April 15, 1956 (p. 126), it was shown that an impedance transformation through a bilateral, two terminal-pair network can be performed by means of a configuration consisting of a line in three-dimensional hyperbolic space and two other lines that are both non-Euclidean perpendicular to this line, which transforms into itself by the transformation. The line will be abbreviated l.t.t.i.i. (line that transforms into itself). The two perpendiculars, which can be arbitrarily shifted along and rotated around the l.t.t.i.i. as long as they keep the same non-Euclidean distance and angle between themselves, yield the multiplier of the canonic form of the bilinear transformation. The fixed points of the transformation correspond to the points at which the l.t.t.i.i. cuts the Riemann sphere. The sphere is shown in Fig. XIV-1.

Let us represent a bilateral two terminal-pair network by a box with an input and an output terminal pair. By performing three measurements (immittance or reflection coefficient measurements) six points are obtained on the Riemann unit sphere. These points are sufficient for determining the l.t.t.i.i. and its perpendiculars (1). The position of the l.t.t.i.i. directly specifies the network.

The following theorems are valid:

1. If the l.t.t.i.i. is imbedded in the yz-plane and the transformation is hyperbolic (the two perpendiculars are in a plane through the l.t.t.i.i.) or if the l.t.t.i.i. is perpendicular to the yz-plane and the transformation is elliptic (the two perpendiculars are in a plane non-Euclidean perpendicular to the l.t.t.i.i.), then the network is lossless. In the transitional case between the two cases mentioned, a parabolic transformation, the l.t.t.i.i. is tangent to the sphere.

2. If the l.t.t.i.i. is imbedded in the xz-plane and the transformation is hyperbolic, or if the l.t.t.i.i. is perpendicular to the xz-plane and the transformation is elliptic, then the network is purely resistive (composed of positive and negative resistances).

3. If the l.t.t.i.i. is perpendicular to the z-axis, then the network is symmetric.

The common method, in splitting a bilateral, two terminal-pair network into lossy and lossless parts is to perform three measurements, having the output terminated in pure reactances. Three points on the great circle in the yz-plane, corresponding to the imaginary axis in the complex impedance plane, are then transformed into three points on the right hemisphere. Through the latter points a circle, the image circle of the great circle, can be drawn.

The first operation is to find a transformation by which the image circle is moved until it is symmetric with the xz-plane. This can be done in several ways, the simplest

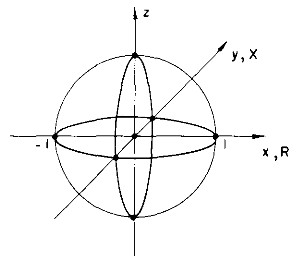


Fig. XIV-1. The Riemann unit sphere.

being the methods of Weissfloch (2) and Wheeler-Dettinger (3). Weissfloch extracts a series reactance, corresponding to a parabolic transformation, having its fixed point at the top of the sphere; Wheeler and Dettinger extract a piece of uniform transmission line, corresponding to an elliptic transformation, having its l.t.t.i.i., for simplicity, coalescing with the x-axis. In Fig. XIV-2 the transformed image circle is shown in the xz-plane as a straight line C.

The second operation is to extract an attenuator so that C is transformed into the projection in the xz-plane of the great circle in the yz-plane. The transformation is hyperbolic. Once more, several methods are possible. Weissfloch (2) utilizes an L-network composed of the series resistance $R_s$ and the shunt resistance $R_p$. In Fig. XIV-2 the l.t.t.i.i., the isometric-circle projections (1), and the points that yield $R_s$ and $R_p$ are graphically constructed (proofs will appear elsewhere). If the attenuator is composed of a symmetric T-network, the corresponding constructions will be those shown in Fig. XIV-3. In Fig. XIV-4 a procedure discussed by Altschuler (4) is shown. He divides the operation into two steps (Fig. XIV-4a and b), a hyperbolic transformation along the z-axis, corresponding to an ideal transformer, and a hyperbolic transformation along the x-axis, corresponding to an ideal reflection coefficient transformer.

The two operations mentioned above have moved the three points on the image circle to the great circle in the yz-plane. In the third operation these points are transformed into the original points on the same great circle by a nonloxodromic transformation, corresponding to a lossless network. This part of the problem has been thoroughly discussed by Van Slooten (5).

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## References

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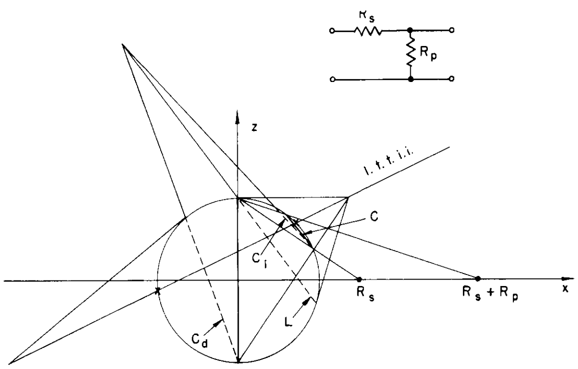


Fig. XIV-2. Transformation through attenuator (L-network).

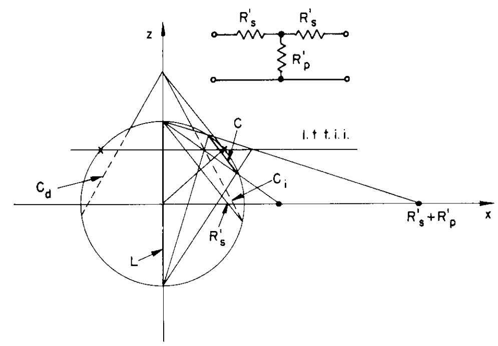


Fig. XIV-3. Transformation through attenuator (symmetric T-network).

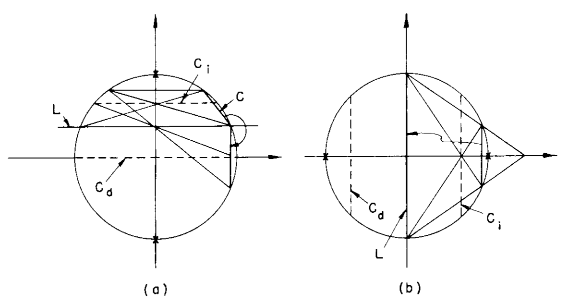


Fig. XIV-4. Transformation through ideal transformer and ideal reflection coefficient transformer.

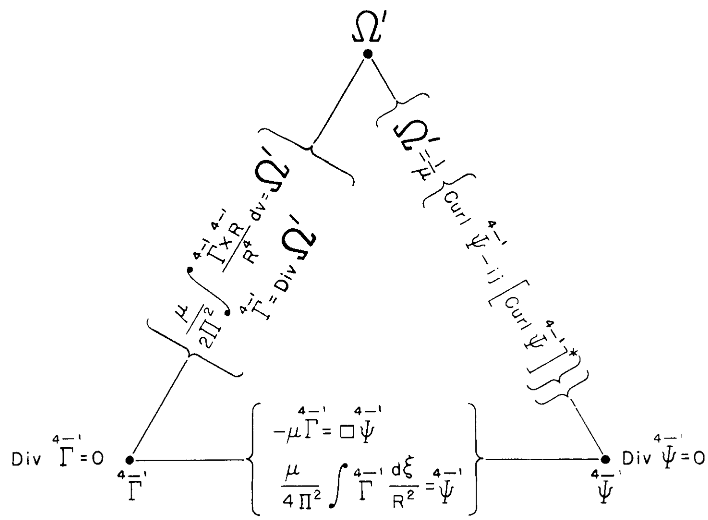


Fig. XIV-5. Classical electromagnetic equations.

## B. USE OF COMPLEX FOUR-DIMENSIONAL QUANTITIES IN ELECTROMAGNETIC WAVE THEORY

In many electromagnetic wave problems involving symmetry between the electric and magnetic modes – for example, in problems dealing with waveguides, cavities, and antennas in the microwave field – it is convenient to introduce a complex vector $\overline{Q'} = a\overline{E} + jb\overline{B}$, where $\overline{E}$ is the electric field strength, $\overline{B}$ is the magnetic induction, $j = \sqrt{-1}$, and a and b are real or pure imaginary constants. (A bar over a letter indicates a vector; an apostrophe, a complex quantity.) The components of the complex vector $\overline{Q'}$ represent two related modes, one electric, and one magnetic. The vector can be used throughout the problem and then finally split into its two parts. Thus the calculation will be simplified and easily examined.

The idea is not new. In 1901 Weber (1) used a quantity $\mathscr{E} + j\mathscr{M}$ ($a = 1$, $b = 1/\mu$) for the purpose of compressing Maxwell's equations. Silberstein introduced different notations in three papers (2-4): $E_1 + jE_2$ ($a = 1$, $b = 1/\mu$); $\sqrt{\epsilon}\, E_1 + j\sqrt{\mu} E_2$ ($a = \sqrt{\epsilon}$, $b = 1/\sqrt{\mu}$), and $\overline{M} - j\overline{E}$ ($a = -j$, $b = -j/\mu$). In all of these cases $\mathscr{E}$, $E_1$, and $\overline{E}$ represent the electric mode; $\mathscr{M}$, $E_2$, and $\overline{M}$ represent the magnetic mode. The dielectric constant is $\epsilon$, and $\mu$ is the permeability. Finally, Stratton (5) gives $\overline{B} + j\sqrt{\epsilon\mu}\overline{E}$ ($a = j\sqrt{\epsilon\mu}$, $b = -j$). Due consideration has to be given to the fact that different writers have used different units in defining electric and magnetic quantities.

There is, however, another formal way of compressing the classical electromagnetic equations. This representation stems from the theory of relativity and utilizes four-dimensional quantities.

If the two methods – the complex-vector method and the four-vector method – are combined, the classical electromagnetic equations can be expressed by means of only three complex, four-dimensional quantities, $\Omega'$, ${}^4\overline{\Gamma}'$, and ${}^4\overline{\Psi}'$, that represent the electromagnetic fields, the currents and charges, and the vector and scalar potentials. See Fig. XIV-5.

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### References

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