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Lindahl's Solution and the Core of An Economy with Public Goods

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LINDAHL'S SOLUTION AND THE CORE OF AN ECONOMY WITH PUBLIC GOODS

I. Introduction

Since Samuelson's clear statement of the theory of consumption externalities (public goods) [7], there has been substantial progress in sharpening analytical tools for tax and expenditure problems. An important part of this progress has been the rehabilitation and reconstruction of Lindahl's [6] quasi-demand solution to the taxation problem by Johansen [4] and Samuelson [8].

More recently the theory of the core of an n-person game has been applied elegantly to the problem of economic general equilibrium [9]. The purpose of this paper is to state rigorously the modern theory of resource allocation with public goods, and to study the relationship between Lindahl's solution and the core.

Section II introduces assumptions and definitions, including the notion of "public competitive equilibrium" of [3]. Section III establishes the relation between marginal rates of transformation and marginal rates of substitution at Pareto optimal allocations of the economy. This is a reconstruction of Samuelson's first paper [7]. Section IV defines and establishes the existence of Lindahl equilibrium. Section V defines the core. Section VI consists of a proof that a Lindahl equilibrium is in the core.

II. Definitions and Notation

The economy has m public goods and k private goods. A vector of public and private goods is written $(x_1, \ldots, x_m; y_1, \ldots, y_k) = (x;y)$.[1]

There are n consumers, distinguished by superscripts.

A. Consumption

Each consumer chooses a point in a consumption set X^i, on which there is defined a complete and transitive preference ordering $\succsim_i$, and owns an initial endowment of private goods w^i.

A.1 $X = \sum_{i=1}^{n} X^i$ has a lower bound for $\leq$

A.2 X^i is closed and convex for each i, and has an interior in the private good subspace.

A.3 If $x^i \in X^i$ there is $\bar{x}^i \in X^i$ with $\bar{x}^i \succ_i x^i$. (Non-satiation.)

A.4 For every $\bar{x}^i \in X^i$ the sets $\{x^i \in X^i \mid x_i \succsim_i \bar{x}_i\}$ and $\{x^i \in X^i \mid x_i \precsim_i \bar{x}_i\}$ are closed in X^i. (Continuity.)

A.5 If $\bar{x}^i \succ_i \hat{x}^i$ then $a\bar{x}^i + b\hat{x}^i \succ_i \bar{x}^i$ where $a,b > 0$ and $a + b = 1$. (Convexity.)

A.6 There is a point $(x;y^i)$ in X^i with $y^i < w^i$.

A.7 If $(x;y) \geq (\bar{x};\bar{y})$, then $(x;y) \succ_i (\bar{x};\bar{y})$ for all i.

B. Production

Production is denoted by a vector $(x;z)$ with inputs negative and outputs positive. The set of all technically possible production plans is called Y.

B.1 Y is a closed, convex cone. That is, if $(\bar{x};\bar{z})$ and $(\hat{x};\hat{y}) \in Y$ and $a,b \geq 0$, then $(a\bar{x} + b\hat{x};\ a\bar{z} + b\hat{z}) \in Y$.

B.2 $0 \in Y$.

B.3 If $(x;z) \neq 0 \in Y$, at least one $z_j < 0$.

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[1] As is customary, $x^1 > x^2$ means $x^1_j > x^2_j$ for all j; $x^1 \geqq x^2$ means $x^1_j \geq x^2_j$ for all j; $x^1 \geq x^2$ means $x^1 \geq x^2$ but not $x^1 = x^2$.

B.4 There is $(x;z) \in Y$ with $x_j > 0$, $j = 1,\ldots,m$. (Possibility of producing public goods.)

B.5 If $(x;z) \in Y$ and $\bar{x}_j = x_j$ when $x_j \geq 0$ but $\bar{x}_j = 0$ when $x_j < 0$, then $(\bar{x};z) \in Y$. (No public good is necessary as a production input.)

C. Allocations

C.1 An <u>allocation</u> is a vector of public goods x and a set of n vectors of private goods $(y^1,\ldots,y^n)$ such that for all i there is $(\bar{x};\bar{y}^i) \in X^i$ with $\bar{y}^i < y^i$.

C.2 A <u>feasible allocation</u> is an allocation $(x;y^1,\ldots,y^n)$ such that $[x;\sum_{i=1}^{n}(y^i-w^i)] \in Y$.

C.3 A <u>Pareto optimum</u> is a feasible allocation $(x;y^1,\ldots,y^n)$ such that there is no other feasible allocation $(\bar{x};\bar{y}^1,\ldots,\bar{y}^n)$ with $(\bar{x};\bar{y}^i) \succ_i (x;y^i)$ for all i.

C.4 A <u>public competitive equilibrium</u> is a feasible allocation $(x;y^1,\ldots,y^n)$, a price system $p = (p_x;p_y)$, and a vector of taxes $(t^1,\ldots,t^n)$ with $p_x \cdot x = \sum_{i=1}^{n} t^i$ such that:

a) $p \cdot [x;\sum_{i=1}^{n}(y^i-w^i)] \geq p \cdot (\bar{x};\bar{z})$ for all $(\bar{x};\bar{z}) \in Y$;

b) $p_y \cdot y^i = p_y \cdot w^i - t^i$ and if $(x;\bar{y}^i) \succ_i (x;y^i)$, then $p_y \cdot \bar{y}^i > p_y \cdot y^i$;

c) there is no vector of public goods and taxes $(\bar{x};\bar{t}^1,\ldots,\bar{t}^n)$ with $p_x \cdot \bar{x} = \sum_{i=1}^{n} \bar{t}^i$ such that for every i there exists $\bar{y}^i$ with $(\bar{x};\bar{y}^i) \succ_i (x;y^i)$ and $p_y \cdot \bar{y}^i \leq p_y \cdot w^i - \bar{t}^i$.

Public competitive equilibrium involves profit maximization by producers, preference maximization under the after-tax budget constraint by consumers, and the impossibility of finding a new public sector with taxes to pay for it that appears to every individual to leave him better off. See Foley [3] for a discussion of this definition and a proof that a public competitive equilibrium is a Pareto optimum.

III. Prices

The prices of public goods in a public competitive equilibrium are marginal rates of transformation. Samuelson [7] has shown that at a Pareto optimum the marginal rates of transformation for public goods are equal to the sums of marginal rates of substitution. This is the essence of the following theorem.

There is a separate public good price vector for each individual. Every individual facing his separate price vector "demands" the same vector of public goods as every other. The sum of the individual price vectors is a social price for the public goods which is the correct guide for profit-maximizing producers to follow. Notice the symmetry: in the case of private goods everyone faces the same prices and each person chooses a different bundle of goods, while in the case of public goods everyone chooses the same bundle but faces different prices.

A. <u>Theorem</u>: If $(x^1; y^1, \ldots, y^n)$ is a Pareto optimum, there exists a price vector $p = (p_x{}^1, \ldots, p_x{}^n; p_y) \geq 0$ such that

a) $(\sum_{i=1}^{n} p_x{}^i; p_y) \cdot [x; \sum_{i=1}^{n}(y^i - w^i)] \geq (\sum_{i=1}^{n} p_x{}^i; p_y) \cdot (\bar{x}; \bar{z})$ for all $(\bar{x}; \bar{z}) \in Y$.

b) If $(\bar{x}; \bar{y}^i) \succ_i (x; y^i)$, then $p_x{}^i \cdot \bar{x} + p_y \cdot \bar{y}^i > p_x{}^i \cdot x + p_y \cdot y^i$.

<u>Proof</u>: Let

$$F = [(x^1, \ldots, x^n; z) \mid x^1 = \ldots = x^n = x \text{ and } (x; z) \in Y].$$

F is a convex cone since if $(\bar{x}^1, \ldots, \bar{x}^n; \bar{z})$ and $(\hat{x}^1, \ldots, \hat{x}^n; \hat{z}) \in F$, and $a, b > 0$, then $(a\bar{x}^1 + b\hat{x}^1, \ldots, a\bar{x}^n + b\hat{x}^n; a\bar{z} + b\hat{z})$ has

$a\bar{x}^1 + b\hat{x}^1 = \ldots = a\bar{x}^n + b\hat{x}^n$ because $\bar{x}^1 = \ldots = \bar{x}^n$ and $\hat{x}^1 = \ldots = \hat{x}^n$, and $(a\bar{x} + b\hat{x};\ a\bar{z} + b\hat{z}) \in Y$ because $(\bar{x};\bar{z})$ and $(\hat{x};\hat{z}) \in Y$ and Y is a convex cone.

Let

$$D = \{(\bar{x}^1,\ldots,\bar{x}^n;\ \bar{z}) \mid \bar{z} = \sum_{i=1}^{n} \bar{z}^i \text{ with } (\bar{x}^i;\ \bar{z}^i + w^i) \succ_i (x;y^i)\}.$$

D is convex since if $(\bar{x}^1,\ldots,\bar{x}^n;\ \bar{z})$ and $(\hat{x}^1,\ldots,\hat{x}^n;\ \hat{z}) \in D$ and $a,b > 0$ with $a + b = 1$, then $a\bar{x}^1 + b\hat{x}^1,\ldots,a\bar{x}^n + b\hat{x}^n;\ a\bar{z} + b\hat{z})$ will have $(a\bar{x}^i + b\hat{x}^i;\ a\bar{z}^i + b\hat{z}^i + w^i) \succ_i (x;y^i)$ by convexity of preference, and so will be in D.

D and F have no points in common, since if they did $(x;\ y^1,\ldots,y^n)$ would not be a Pareto optimum. By the Minkowski separating hyperplane theorem (for one proof see Karlin [5]), there exists a price vector $(p_x^1,\ldots,p_x^n;\ p_y) \neq 0$ and a scalar r such that

(1) for all $(\bar{x}^1,\ldots,\bar{x}^n;\ \bar{z}) \in F$, $\sum_{i=1}^{n} p_x^i \cdot \bar{x} + p_y \cdot \bar{z} \leq r$; and

(2) for all $(\bar{x}^1,\ldots,\bar{x}^n;\ \bar{z}) \in D$, $\sum_{i=1}^{n} p_x^i \cdot \bar{x}^i + p_y \cdot \bar{z} \geq r$.

By continuity of preference $(x,\ldots,x;z) \in \bar{D}$, the closure of D, and is in F so that

$$\sum_{i=1}^{n} p_x^i \cdot x + p_y \cdot z = r \geq \sum_{i=1}^{n} p_x^i \cdot \bar{x} + p_y \cdot \bar{z}$$

for all $(\bar{x};\bar{z}) \in Y$, which establishes part a) of the theorem.

Since $0 \in F$, $r \geq 0$ and if r were positive the activity $(x,\ldots,x;z)$ could be expanded indefinitely giving a higher profit and contradicting (1) so that $r = 0$.

Suppose p had some negative component. Points with very large amounts of the corresponding commodity would be in $\bar{D}$ by monotonicity, but would have smaller value than $(x,\ldots,x;z)$ which contradicts (2). Therefore $p \geq 0$.

Suppose, however, that $p_y = 0$. Some $p_x^h \geq 0$. Since production of all public goods is possible with no public good input, there would be a point in F with positive profit, which contradicts (2). Therefore $p_y \geq 0$.

Suppose $(\bar{x}^h; \bar{y}^h) \succ_i (x^h; y^h)$. The point $[\bar{x}^1, \ldots, \bar{x}^n; \sum_{i=1}^{n}(\bar{y}^i - w^i)]$ where $\bar{y}^k = y^k$ and $\bar{x}^k = x^k$ for $k \neq h$ is in $\bar{D}$ so

$$\sum_{i=1}^{n} p_x^i \cdot \bar{x}^i + p_y \cdot [\sum_{i=1}^{n}(\bar{y}^i - w^i)] \geq \sum_{i=1}^{n} p_x^i \cdot x + p_y \cdot [\sum_{i=1}^{n}(y^i - w^i)].$$

Since all terms are the same on both sides except those corresponding to h, it must be true that

$$p_x^h \cdot \bar{x}^h + p_y \cdot \bar{y}^h \geq p_x^h \cdot x + p_y \cdot y^k.$$

Suppose equality held. Since there is in X^i another point with all private goods smaller and $p_y \geq 0$, there is another point in X^i with lower value. Along the line between this point and $(\bar{x}^i; \bar{y}^i)$ all points have lower value than $(x; y^i)$, but near $(\bar{x}^i; \bar{y}^i)$ by continuity there will be a point preferred to $(x; y^i)$. This corresponds to a point in $\bar{D}$ with smaller value than $(x, \ldots, x; z)$ which contradicts (2). Therefore

$$p_x^h \cdot \bar{x}^h + p_y \cdot \bar{y}^h > p_x^h \cdot x + p_y \cdot y^h$$

and part b) of the theorem is established.

I leave it to the reader to verify that $p_x = \sum_{i=1}^{n} p_x^i$ and $t^i = p_y \cdot (w^i - y^i)$ also satisfy the definition of public competitive equilibrium in II.C.4.

IV. Lindahl Equilibrium

Corresponding to each Pareto optimum there is a total tax on each person

$$t^i = p_y \cdot (w^i - y^i)$$

which may be positive or negative. There is also a value for total consumption

$$p_x^i \cdot x + p_y \cdot y^i$$

which may be greater or smaller than endowment income. Samuelson [8] proposes to call the difference between this consumption value and endowment income a lump-sum transfer:

$$L^i = p_x^i \cdot x + p_y \cdot y^i - p_y \cdot w^i.$$

As Johnson [4] explicated clearly, it was Lindahl's idea [6] to find a kind of mock competitive equilibrium in the public sector. On the assumption that the endowments of the society have already been redistributed to achieve an ethical optimum, this competitive equilibrium will also be the social welfare optimum.

Competitive equilibrium requires that all consumers be maximizing satisfaction at the income they get from their own endowment. In other words all the lump-sum transfers must be zero. In the language of the "benefit theory," the value of public goods received by each individual is equal to the total tax he pays.

A. Definition

A Lindahl equilibrium is a feasible allocation $(x; y^1, \ldots, y^n)$ and a price system $(p_x^1, \ldots, p_x^n; p_y) \geq 0$ such that

a) $(\sum_{i=1}^{n} p_x^i;\ p_y) \cdot [x; \sum_{i=1}^{n} (y^i - w^i)] \geq (\sum_{i=1}^{n} p_x^i;\ p_y) \cdot (\bar{x}; \bar{z})$ for all $(\bar{x}; \bar{z}) \in Y$;

b) if $(\bar{x}^i; \bar{y}^i) \succ_i (x; y^i)$ then $p_x^i \cdot \bar{x}^i + p_y \cdot \bar{y}^i > p_x^i \cdot x + p_y \cdot y^i = p_y \cdot w^i$.

B. Existence

To prove the existence of a Lindahl equilibrium the following lemma, adapted trivially from Debreu [2] is necessary.

Lemma: Given the assumptions II.A.1-5 and II.B.1-3, there exists in a private good economy a feasible allocation $(y^1,\ldots,y^n)$ and a price vector $p \neq 0$ such that

a) $p\cdot[\sum_{i=1}^{n}(y^i - w^i)] \geq p\cdot\bar{z}$ for all $\bar{z} \in Y$;

b) for every i, if $\bar{y}^i \succ_i y^i$ then $p_y\cdot\bar{y}^i \geq p_y\cdot y^i = p_y\cdot w^i$.

Theorem: Under assumptions II.A and II.B there exists a Lindahl equilibrium.

Proof: The strategy is to construct a private good economy to which the lemma applies and show that the quasi-equilibrium of this economy is a Lindahl equilibrium.

Extend the commodity space by considering each consumer's bundle of public goods as a separate group of commodities. In this nm + k space, the set F meets the requirements of II.B.1-3.

Extend the sets X^i by writing zeroes for all public good components not corresponding to the i^{th} consumer. These sets meet the requirements of II.A.1-5.

From the Lemma, there will be a vector $(x^1,\ldots,x^n;\ y^1,\ldots,y^n)$ and prices $(p_x^1,\ldots,p_x^n;\ p_y) \neq 0$ such that

(1) $(\sum_{i=1}^{n}p_x^i;\ p_y)\cdot[x;\ \sum_{i=1}^{n}(y^i - w^i)] \geq (\sum_{i=1}^{n}p_x^i;\ p_y)\cdot(\bar{x};\bar{z})$ for all $(\bar{x};\bar{z})$ in Y. ($x^1 = \ldots = x^n = x$ as a consequence of the definition of F.)

(2) if $(\bar{x}^i;\ \bar{y}^i) \succ_i (x;y^i)$ then

$$p_x^i\cdot\bar{x}^i + p_y\cdot\bar{y}^i \geq p_x^i\cdot x + p_y\cdot y^i = p_y\cdot w^i.$$

An argument similar to the one based on monotonicity used in section II will assure that $p \geq 0$ and $p_y \geq 0$.

Part (2) implies

(3) if $(\bar{x}^i; \bar{y}^i) \succ_i (x;y^i)$ then

$$p_x^i \cdot \bar{x}^i + p_y \cdot \bar{y}^i \geq p_x^i \cdot x + p_y \cdot y^i = p_y \cdot w^i.$$

Suppose equality held. Since there is in X^i a point with all private goods smaller than w^i, and $p_y \geq 0$, there is a point in X^i with smaller value than $(x;y^i)$. This will lead to a contradiction (as in the theorem of section II), so that

(4) if $(\bar{x}^i; \bar{y}^i) \succ_i (x;y^i)$ then

$$p_x^i \cdot \bar{x}^i + p_y \cdot \bar{y}^i > p_x^i \cdot x + p_y \cdot y^i = p_y \cdot w^i.$$

This establishes the theorem.

V. The Core

A. Definitions

An allocation $(x; y^1,\ldots,y^n)$ is said to be <u>blocked</u> by a subset or coalition of consumers S if there exists $(\bar{x}; \bar{y}^1,\ldots,\bar{y}^n)$ with $\bar{x} \geq 0$, and $(\bar{x}; \bar{y}^i) \succ_i (x;y^i)$ for all $i \in S$ such that $[\bar{x}; \sum_{i \in S}(\bar{y}^i - w^i)] \in Y$.

Notice that the $\bar{y}^i$ corresponding to consumers not in S may be zero.

An allocation is in the <u>core</u> of the economy if it cannot be blocked by any coalition. If an allocation is in the core, no coalition can do better for all its members with its own resources.

VI. A Lindahl Equilibrium is in the Core

A. <u>Theorem</u>: If $(x; y^1,\ldots,y^n; p_x^1,\ldots,p_x^n; p_y)$ is a Lindahl equilibrium it is in the core.

<u>Proof</u>: Suppose the coalition S could block $(x; y^1,\ldots,y^n)$ by $(\bar{x}; \bar{y}^1,\ldots,\bar{y}^n)$. Since $(\bar{x};\bar{y}^i) \succ_i (x;y^i)$ for all $i \in S$, we know by the

definition of Lindahl equilibrium that

$$(1)\quad \sum_{i\varepsilon S} p_x^{\,i}\cdot\bar{x} + p_y\cdot\sum_{i\varepsilon S}\bar{y}^i > \sum_{i\varepsilon S} p_x^{\,i}\cdot x + p_y\cdot\sum_{i\varepsilon S} y^i = p_y\cdot\sum_{i\varepsilon S} w^i.$$

Since $p_x^{\,i} \geq 0$ for all i, $\sum_{i=1}^{n} p_x^{\,i} \geq \sum_{i\varepsilon S} p_x^{\,i}$, which means that, since $\bar{x} \geq 0$,

$$(2)\quad \sum_{i=1}^{n} p_x^{\,i}\cdot\bar{x} + p_y\cdot\sum_{i\varepsilon S}(\bar{y}^i - w^i) > 0.$$

But the profit maximizing condition for Lindahl equilibrium asserts that

$$(3)\quad \sum_{i=1}^{n} p_x^{\,i}\cdot\bar{x} + p_y\cdot\bar{z} \leq 0 \text{ for all } (\bar{x};\bar{z})\ \varepsilon\ Y.$$

This contradiction establishes the theorem.

VII. Conclusion

There is no reason to think that Lindahl equilibrium can be embodied by any working political process. There is some reason to think that the core is a meaningful political concept. If a group of people find themselves able, using only their own resources, to achieve a better life, it is not unreasonable to suppose that they will try to enforce this threat against the rest of the community. They may find themselves frustrated if the rest of the community resorts to violence or force to prevent them from withdrawing. In the absence of some assumption about the reaction of the part of the society not in the coalition to the coalition threat, detailed predictions of the equilibrium situation are not possible.

But if a society stays inside the core as it is defined here, there is a minimal rationale for everyone to continue to participate. The conflicts that will naturally arise over the redistribution of initial endowments will still be there. But no group will have the power to alter the situation in its own favor unilaterally. This is a very crude and intuitive argument for the relevance of the core to political reality, but the whole theory of political equilibrium is in its infancy.

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